Spectral method for inhomogeneous superconducting strip problems and magnetic flux pump modeling

Leonid Prigozhin[1] & **Vladimir Sokolovsky**[2]

[1]Blaustein Institutes for Desert Research and [2]Physics Department

Ben-Gurion University of the Negev

8-th Int. Workshop on Modelling in HTS, Nancy, June 14-16, 2022

Alternative approaches to numerical solution of superconductivity problems

1. FFT-based methods:

- Derived for thin film problems (Vestgården et al. 2012, 2013);
- Improved and extended to 3D bulk and stack problems (LP and Sokolovsky, 2018).

2. Spectral methods (Sokolovsky and LP, 2020-2022):

- Chebyshev solution of singular 1D integral equations and their systems: homogeneous sc strips, stacks, pancake coils;
- Hermite-Chebyshev solution of 2D inhomogeneous strip problems.

Problem formulation :

Let a thin strip of the width $2a$ be presented by a 2D domain in the $z=0$ plane: $\Omega=\{(x,y)\,|\,|x|<\infty, |y|\le a\}$. By the Faraday law

$$\mu_0\dot{h}_z=-\nabla\times\boldsymbol{e},$$

where $\boldsymbol{e}$ is the parallel to strip electric field component, $\nabla\times\boldsymbol{e}=\partial_x e_y-\partial_y e_x$.

We assume $\boldsymbol{e}=\rho\boldsymbol{j}$ with, e.g., a power law for the nonlinear resistivity,

$$\rho=e_0(|\boldsymbol{j}|/j_c)^{n-1}/j_c.$$

The critical sheet current density j_c can depend on the magnetic field $\boldsymbol{h}$ and be also spatially inhomogeneous. The external magnetic field in the strip plane, $\boldsymbol{h}^{\mathrm{e}}(x,y,t)$, can be spatially nonuniform too.

We assume the inhomogeneities are localized: as $|x|\to\infty$

$$\boldsymbol{h}^{\mathrm{e}}(x,y,t)\to\boldsymbol{h}^{\mathrm{e}}_{\infty}(t) \text{ and } j_c(x,y,\boldsymbol{h})\to j_{c,\infty}(\boldsymbol{h}).$$

Stream function reformulation

For simplicity, the derivation is for $I(t)=0,\ j_c$=const, $\boldsymbol{h}^{\mathrm{e}}\to 0$ as $|x|\to\infty$.

Introducing the stream (magnetization) function, $\bar{\nabla}\times g=\boldsymbol{j},\ g|_{y=\pm a}=0,$

and using the Green function $G=(4\pi|\boldsymbol{r}|)^{-1}$ to express the magnetic field induced by the strip current, we arrive at the following formulation:

$$-\nabla\times\int_{\Omega}G(\boldsymbol{r}-\boldsymbol{r}')\bar{\nabla}'\times\dot{g}(\boldsymbol{r}',t)\,\mathrm{d}\boldsymbol{r}'=\dot{h}_z^{\mathrm{e}}+\mu_0^{-1}\nabla\times\boldsymbol{e},$$

$$\boldsymbol{e}=e_0(|\bar{\nabla}\times g|/j_c)^{n-1}\bar{\nabla}\times g/j_c,$$

$$g|_{y=\pm a}=0,\quad g|_{t=0}=g_0,$$

where $\boldsymbol{r}=(x,y)$. For the dynamo pump problem such eqs. were solved by the FFT-based method and two f.e. methods (LP and Sokolovsky 2021; Ghabeli, Pardo, Kapolka 2021).

Method of lines with Hermite - Chebyshev approximation in space

At each moment in time, knowing g we need to find $\dot{g}$ to proceed further.

$$-\nabla \times \int_{\Omega} G(\boldsymbol{r}-\boldsymbol{r}')\bar{\nabla}' \times \dot{g}(\boldsymbol{r}',t)\,\mathrm{d}\boldsymbol{r}' = \dot{h}_z^{\mathrm{e}} + \mu_0^{-1}\nabla \times \boldsymbol{e},$$

$$\boldsymbol{e} = e_0(|\bar{\nabla} \times g|/j_c)^{n-1}\bar{\nabla} \times g/j_c, \qquad g|_{y=\pm a} = 0.$$

The problem for $\dot{g}$ is a singular 2D integro-differential equation.

The 1D Fredholm integral eqs with the Cauchy or logarithmic singularities can be very efficiently solved by the Chebyshev spectral methods.

Our 2D equation with the Green-function-related kernel is more complicated; a new approach was needed.

Hermite - Chebyshev approximation in space

First, we apply the Fourier transform with respect to x (along the strip),

$\tilde{f}(k,y,t) = F[f(x,y,t)]$, and use the convolution theorem.

In dimensionless variables

$$\hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{a}, \quad \hat{\boldsymbol{j}} = \frac{\boldsymbol{j}}{j_c}, \quad \hat{\boldsymbol{h}} = \frac{\boldsymbol{h}}{j_c}, \quad \hat{I} = \frac{I}{aj_c}, \quad \hat{g} = \frac{g}{aj_c}, \quad \hat{\boldsymbol{e}} = \frac{\boldsymbol{e}}{e_0}, \quad \hat{t} = \frac{e_0}{a\mu_0 j_c} t$$

this yields, for each wave number k, a 1D integro-diff. eq. for $\dot{\tilde{g}}(k,y,t)$:

$$-k^2 \int_{-1}^{1} \tilde{G}(k, y-y')\dot{\tilde{g}}(k,y',t)\mathrm{d}y' + \int_{-1}^{1} \partial_y \tilde{G}(k, y-y')\partial_{y'}\dot{\tilde{g}}(k,y',t)\mathrm{d}y' = \tilde{Z},$$

where $Z = \dot{h}_z^{\mathrm{e}} + \partial_x e_y - \partial_y e_x$. The kernels contain the modified Bessel functions,

$$\tilde{G}(k,y) = (2\pi)^{-1} K_0\left(|yk|\right) \quad \text{and} \quad \partial_y \tilde{G}(k,y) = -(2\pi)^{-1}|k| K_1\left(|yk|\right)\mathrm{sign}(y),$$

and are singular: for $s \to 0$ we have $K_0(|s|) \simeq -\ln(|s|/2), \quad K_1(|s|) \simeq 1/|s|$.

Hermite - Chebyshev approximation in space

Singling out the singularities, for each wave number we obtain

$$k^2 \int_{-1}^{1} \left\{ \mathcal{A}(k, y-y') - \frac{1}{2\pi} \ln\left(\frac{|k|}{2} |y-y'| \right) \right\} \dot{\tilde{g}}(k, y', t)\, \mathrm{d}y' +$$

$$\int_{-1}^{1} \left\{ \mathcal{B}(k, y-y') + \frac{1}{2\pi} \frac{1}{y-y'} \right\} \partial_{y'} \dot{\tilde{g}}(k, y', t)\, \mathrm{d}y' = -\tilde{Z},$$

where the functions $\mathcal{A}$ and $\mathcal{B}$ are regular. We now seek

$$\dot{\tilde{g}}(k, y, t) = \sqrt{1-y^2} \sum_{m=0}^{M} \alpha_m(t, k) U_m(y).$$

where U_m are Chebyshev polynomials of the second type. Why in this form?

a) the boundary conditions $g|_{y=\pm 1} = 0$ hold automatically;

b) a very convenient analytical treatment of the singular terms.

Hermite - Chebyshev approximation in space

For $\int_{-1}^{1}\left\{\frac{1}{y-y'}\right\}\partial_{y'}\tilde{\dot{g}}(k,y',t)\mathrm{d}y'$ we find the expansion derivative using

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[U_m(y)\sqrt{1-y^2}\right]=-\frac{(m+1)T_{m+1}(y)}{\sqrt{1-y^2}},$$

where $T_m(y)$ are the Chebyshev polynomials of the first type, then employ

$$\int_{-1}^{1}\frac{T_{m+1}(y')}{(y-y')\sqrt{1-y'^2}}\mathrm{d}y'=-\pi U_m(y).$$

The logarithmic singularity in $\int_{-1}^{1}\ln(|y-y'|)\tilde{\dot{g}}(k,y',t)\mathrm{d}y'$ is also treated analytically. Interpolating double Chebyshev expansions of $\mathcal{A}$ and $\mathcal{B}$ are used to compute the convolutions with these regular kernels.

Hermite - Chebyshev approximation in space

Knowing $g(x,y,t)$ for time t, we compute $\boldsymbol{j}$, $\boldsymbol{e}$, then the r.h.s. of our eq.,

$Z = \dot{h}_{\mathrm{z}}^{\mathrm{e}} + \partial_x e_y - \partial_y e_x$, and approximate its Fourie image $\tilde{Z} = F(Z)$

by a Chebyshev expansion. This enables us to find the coefficients

$\alpha_m(t,k)$ in $\tilde{\dot{g}}(k,y,t) = \sqrt{1-y^2}\sum_{m=0}^{M}\alpha_m(t,k)U_m(y)$

and it remains to return to the real space, $\dot{g} = F^{-1}(\tilde{\dot{g}})$.

This scheme needs an efficient numerical procedure for the Fourier transform and its inverse.

One can replace the strip by a finite one and use the FFT on a uniform grid.

Here we explored an alternative approach based upon Hermite functions.

Hermite - Chebyshev approximation in space

Hermite functions are expressed via Hermite polynomials H_j as

$$\Psi_j(x) = \left(\pi^{1/4}\sqrt{2^j j!}\right)^{-1} e^{-x^2/2} H_j(x),$$

satisfy the orthogonality relation $\int_{-\infty}^{\infty} \Psi_i \Psi_j \mathrm{d}x = \delta_{ij}$, and the recurrent relation

$$\Psi_{j+1}(x) = x\sqrt{\frac{2}{j+1}}\Psi_j(x) - \sqrt{\frac{j}{j+1}}\Psi_{j-1}(x).$$

Since $F[\Psi_j(x)] = \sqrt{2\pi}(-\mathrm{i})^j \Psi_j(k),$ computing Fourier transform and its inverse for expansions in Hermite functions is trivial.

For problems on the infinite axis, scaled basis functions $\Psi_j(x/L)$ are used to better approximate solution in its area of variation; in our case this area is a few times larger than the inhomogeneity domain.

Hermite - Chebyshev method : general scheme

1. The main variables are the grid values $g_{n,m}(t) = g(Lx_n, y_m, t)$, where x_n and y_m are the roots of H_{N+1} and the Chebyshev points $\cos(\pi m/M)$, resp.

2. Double Hermite-Chebyshev interpolating expansions of the grid values are used for numerical differentiation, Fourier transform, etc.

3. For problems with a field dependent j_c magnetic field is efficiently computed using the double Hermite-Chebyshev expansions.

4. Problems with a nonzero transport current $I(t)$ and $h^e \to h^{e,\infty}(t) \neq 0$ at infinity: a similar scheme for $g^* = g - g^{1D}$, where g^{1D} is the solution of a 1D problem for a homogenous strip and the same $I(t)$, $\boldsymbol{h}^{e,\infty}$, computed using the Chebyshev spectral method.

5. Integration in time: Matlab ODE solver ode15s.

Example 1. Inhomogeneous strip with transport current

Let $2a = 12$ mm, $I = 200\sin(2\pi t/T)$ A, $T = 0.02$ s, $\boldsymbol{h}^{\mathrm{e}} = 0$, $n = 20$.

The critical sheet current density

$$j_c = j_c^0 \chi^r(\mathbf{r})\chi^h(h_z),$$

where

$$j_c^0 = 23.6 \text{ A/mm},$$

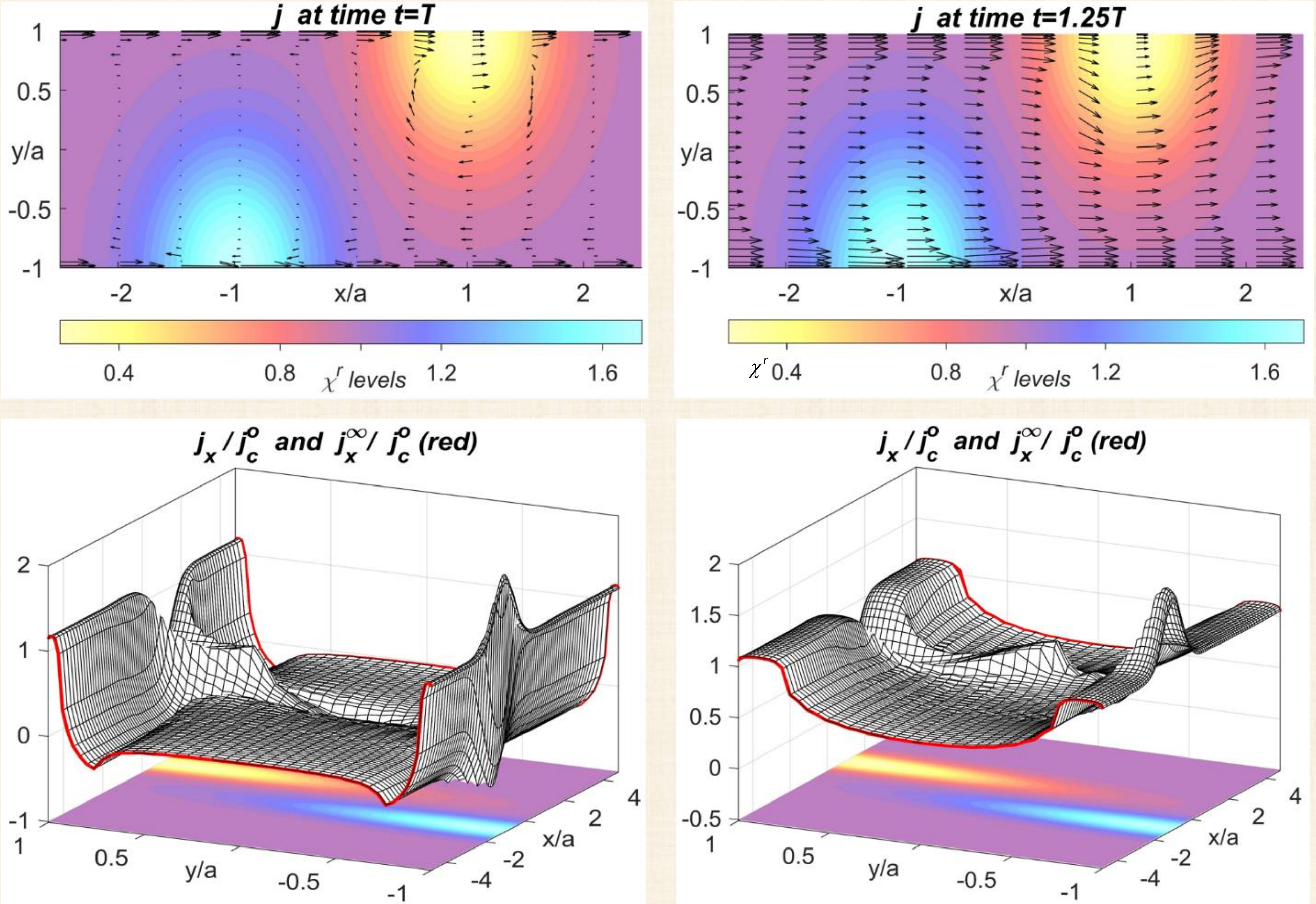
$\chi^h = 1/(1 + h_0^{-1}|h_z|)$ - dependence on the magnetic field with $h_0 = 20\, j_c^0$,

$\chi^r = 1 + \varphi(x+a, y+a) - \varphi(x-a, y-a)$ - spatial inhomogeneity with

$$\varphi(x,y) = 0.75\exp\left(-\left[2x^2 + y^2\right]/a^2\right),$$

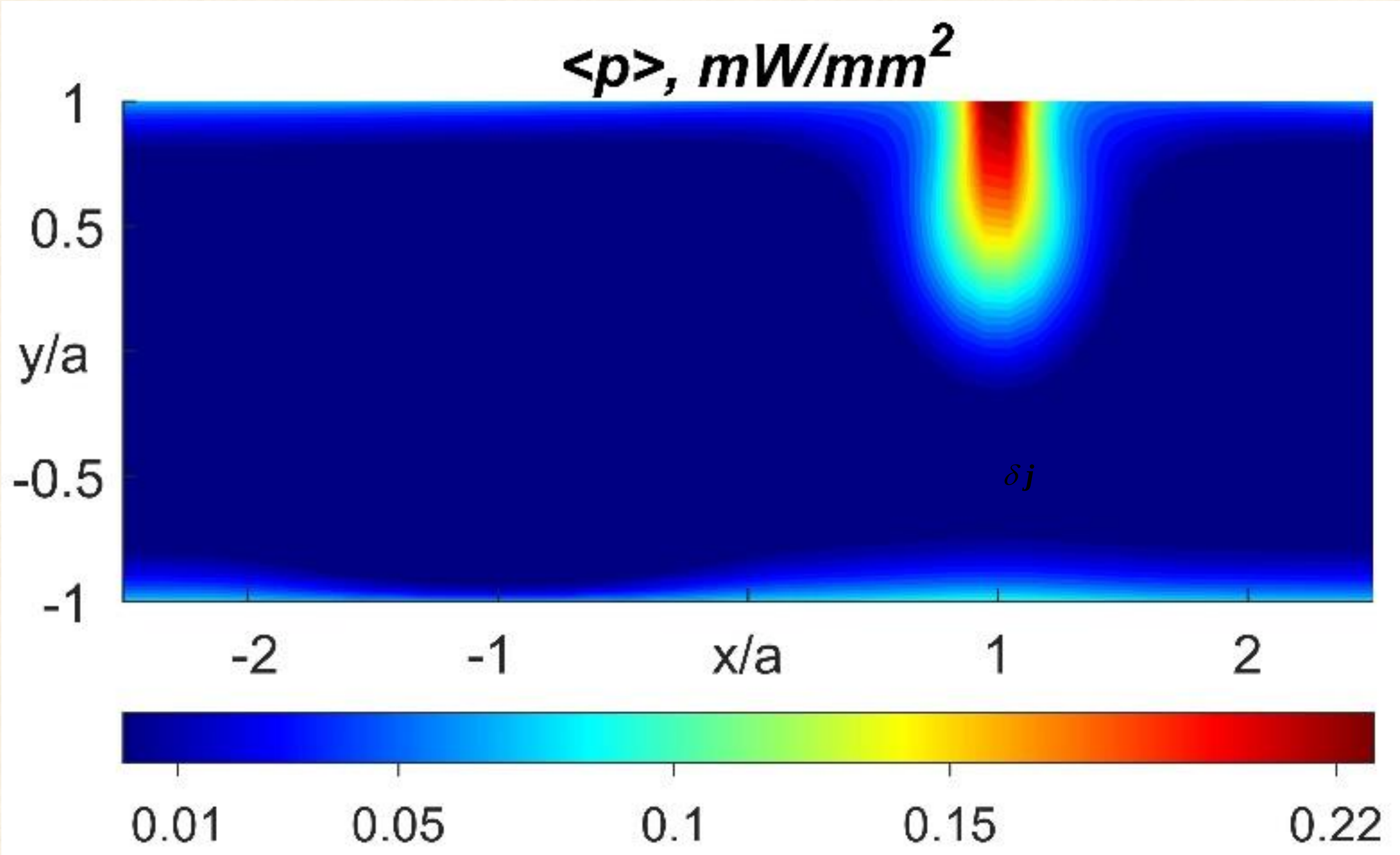
i.e. j_c is higher near $(-a, -a)$ and lower near (a, a) for the same h_z.

Example 1. Inhomogeneous strip, simulation results



- Sheet current density for *I(T)*=0 and *I(1.25T)*=200 A.
- Background color indicates the spatial inhomogeneity of the strip.
- Away from the strip inhomogeneity the sheet current density is close to the solution of the 1D problem for the homogeneous strip (red lines).

Example 1. Inhomogeneous strip, simulation results

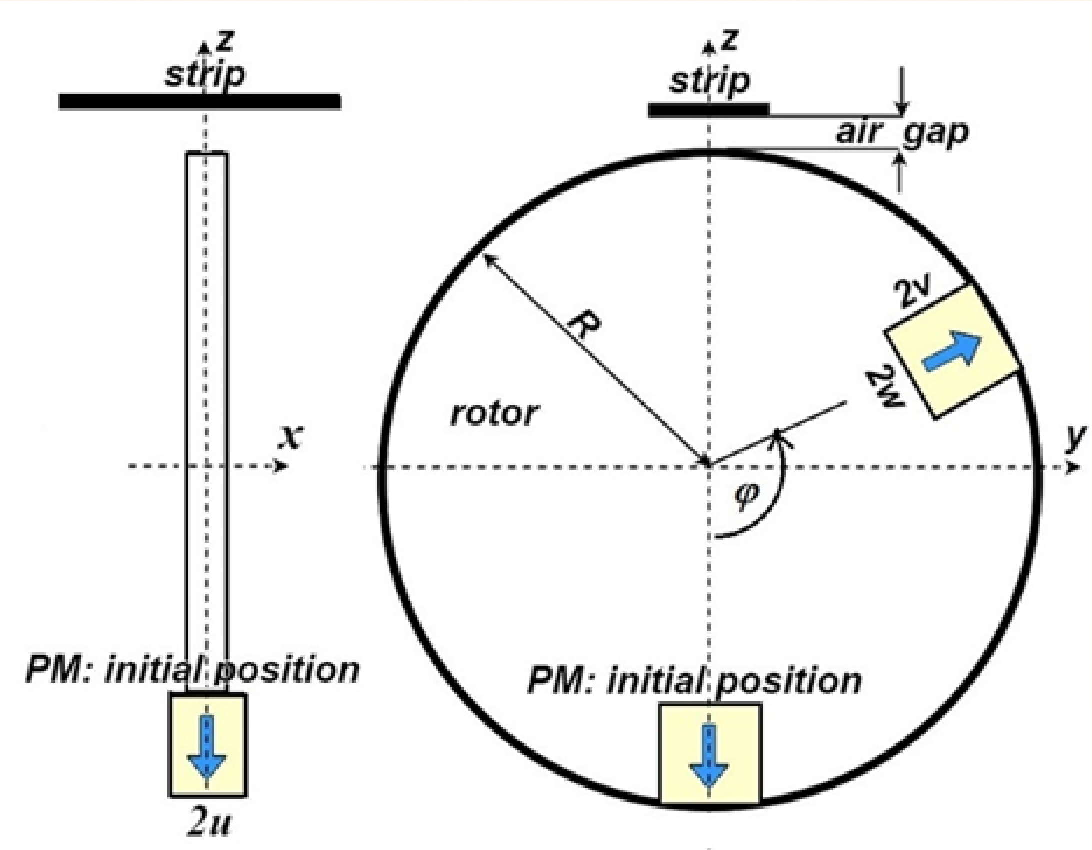


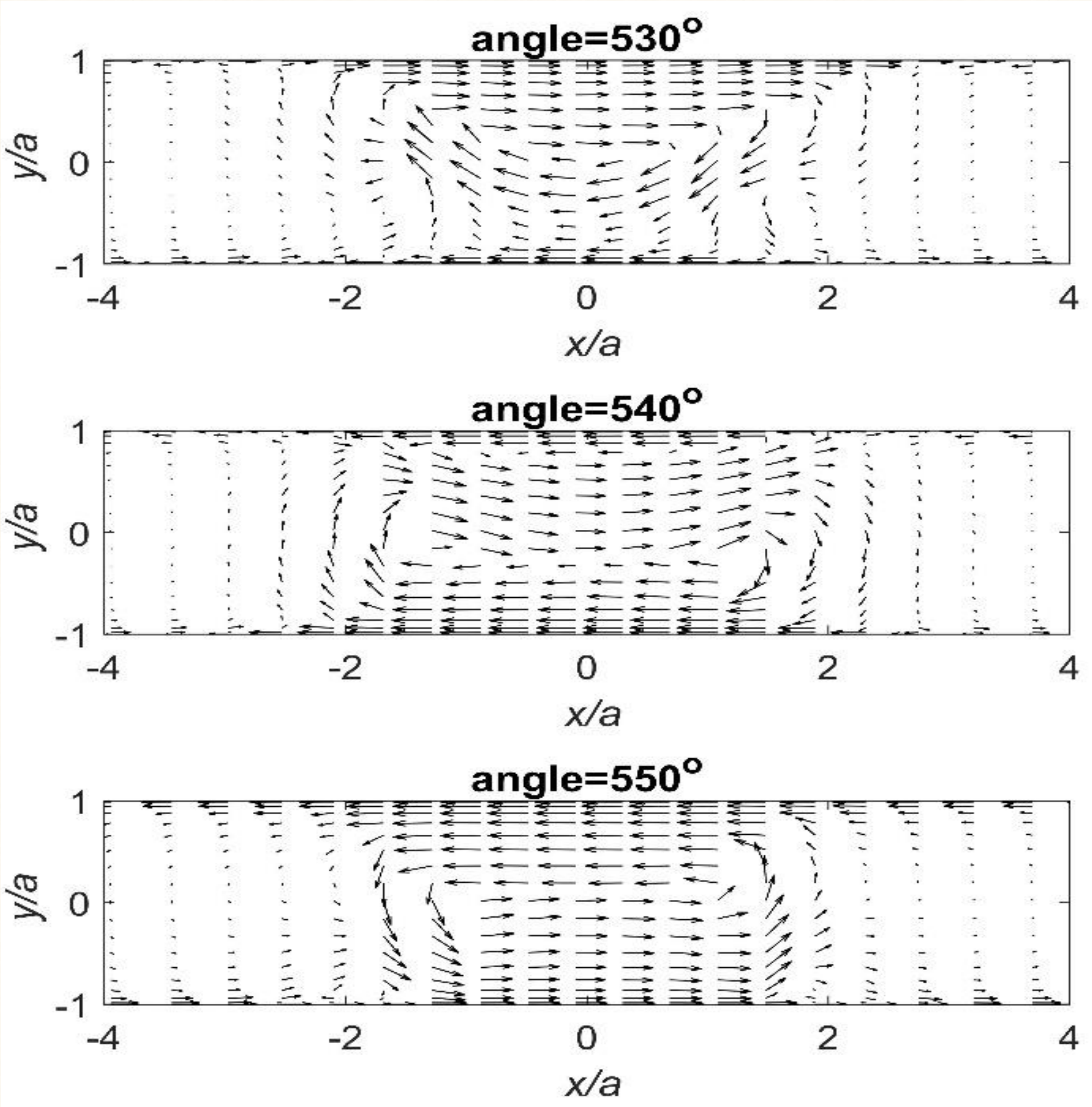
The time-averaged loss power density <p>, high in the lower sheet critical current density area.

Convergence

Mesh			CPU time per cycle	δj (%)
M	N	L (scaling)		
25	40	0.5	36 s	5.1
50	80	0.4	15 m	1.6
100	160	0.3	14 h	--

Example 2. Strip in a nonuniform field : Dynamo flux pump





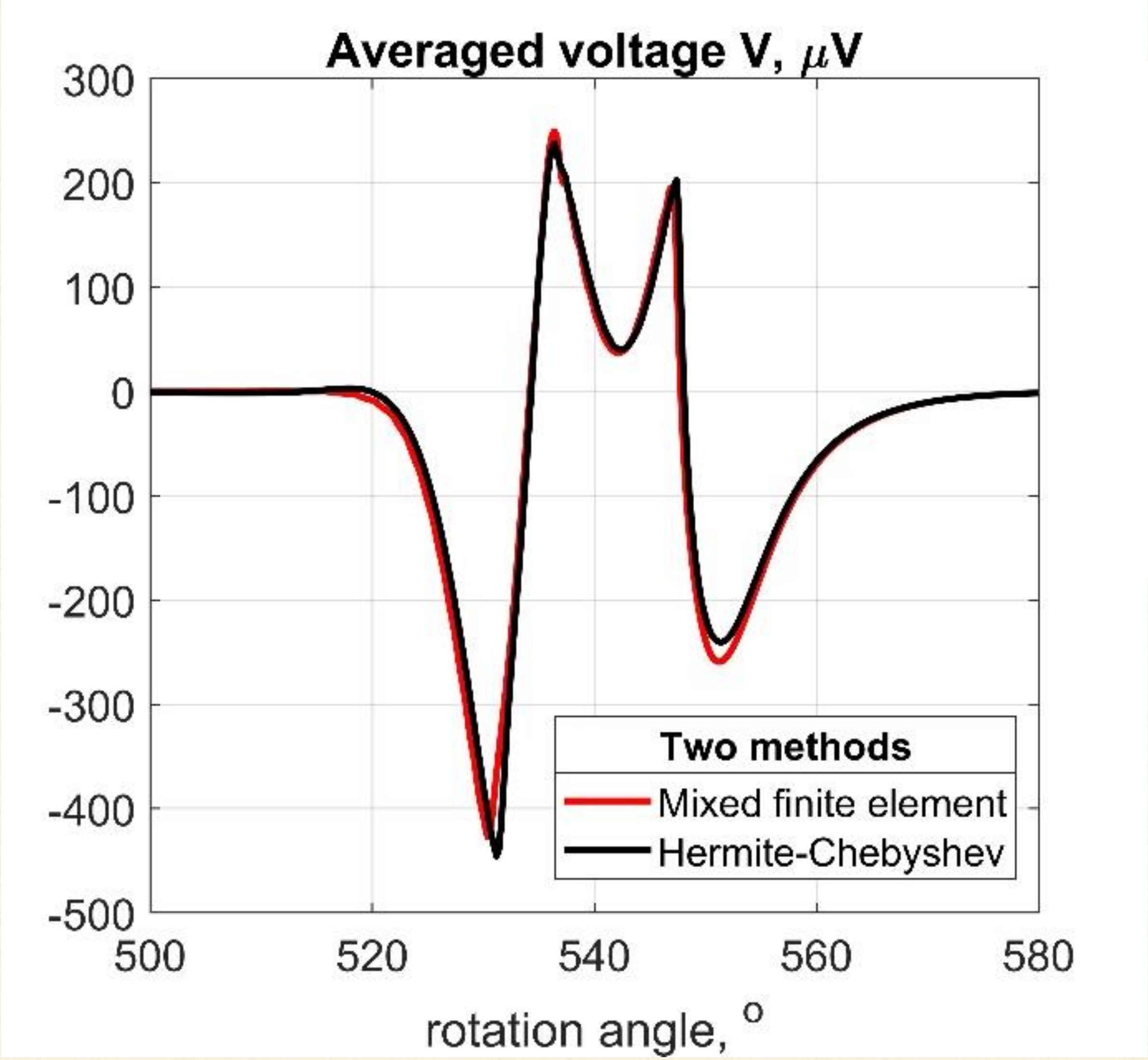
Pump parameters: as in the simplified 1D benchmark problem (Ainslie *et al*, SuST 2020).

We performed a full 2D simulation for the open-circuit condition (I=0) assuming

$$j_c = j_c^0 / \left(1 + h_0^{-1}\sqrt{h_z^2 + 0.5\left[h_x^2 + h_y^2\right]}\right)$$

Computed sheet current density for three rotor positions, $h_0 / j_c^0 = 10$

Example 2. Strip in a nonuniform field : Dynamo flux pump



$$V(t) = (2a)^{-1} \int_{\Omega} e_x(\boldsymbol{r}, t) \mathrm{d}\boldsymbol{r}.$$

Table. Convergence of the two methods.

Mixed finite element method			Hermite-Chebyshev method				
Mesh, number of elements	Time per cycle (min)	δV, %	Mesh			Time per cycle (min)	δV, %
			M	N	L		
1056	3.4	4.5	60	30	0.7	3.3	2.4
2180	33	1.6	90	46	0.6	31	0.9
4226	118	---	120	60	0.5	186	---

Comparison with the mixed f. e. method (LP & Sokolovsky, 2021); $h_0 = \infty$.

Although its convergence is not exponential, the Hermite-Chebyshev method is efficient and fully competitive with the mixed f.e. method.

Conclusion:

We developed the Hermite-Chebyshev method, applicable to inhomogeneous superconducting strip problems with an arbitrary current-voltage relation. This method can be more efficient than the mixed f.e. and the FFT-based methods.

Reference: Sokolovsky and Prigozhin, SuST (2022) **35,** 024002

Thank you for your attention!