Spectral method for inhomogeneous superconducting strip problems and magnetic flux pump modeling

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***Abstract*—We present a new spectral method for solving 2D magnetization and transport current superconducting strip problems with an arbitrary current-voltage relation, spatially inhomogeneous strips, and strips in a nonuniform applied field. The method is based on bivariate expansions in Chebyshev polynomials and Hermite functions and can be employed for numerical modeling magnetic flux pumps of different types and investigating AC losses in coated conductors with local defects. Using a realistic 2D version of the superconducting dynamo benchmark problem as an example, we show that this spectral method is a competitive alternative to finite element methods.**

Keywords—spectral method, high-T_c superconducting dynamo, inhomogeneous superconducting strip.

I. Introduction

Although finite element methods remain the most popular and versatile tool for solution of applied superconductivity problems, alternative approaches have also been proposed recently. One of them is the FFT-based methods for the 2D thin film and 3D bulk problems [1,2], another is based on the Chebyshev spectral methods derived for the problems formulated as a 1D integrodifferential equation with a Cauchy-type singular kernel or as a system of such equations (superconducting strips with or without transport current in a uniform field, stacks of strips, pancake coils) [3]. Ideally, the spectral methods demonstrate an extremely fast exponential convergence. Even if this is not achieved, e.g., if the solution is not smooth enough, etc., they can still be at least as efficient as the finite element methods. Here we extended the spectral method to the 2D thin strip problems.

While Chebyshev polynomial expansions are convenient for solving 1D integral equations with the Cauchy and logarithmic type singular kernels, no similar approach has been developed for the Green-function-based 2D integral formulation. To circumvent this difficulty, we applied the Fourier transform in the along-strip direction. For each wave number, the 2D integral equation is represented in the Fourier space by a 1D singular integral equation written for the transverse coordinate; expansions in Chebyshev polynomials can be used to efficiently solve these equations. To realize this approach, one needs an efficient numerical implementation of the direct and inverse Fourier transforms. Typically, this is realized by replacing the infinite domain by a sufficiently large finite one and using the FFT algorithm on a uniform mesh. A different method we explored in our work is based on expansions in the Hermite functions which diagonalize the Fourier transform. An approximate solution to the 2D integrodifferential problem is, therefore, sought at each moment in time using bivariate Hermite-Chebyshev expansions. The method of lines is employed for integration in time.

II. Problem Formulation

Let the superconducting strip of the width $2a$ be represented by a 2D domain in the plane $z=0$: $\Omega=\left\{\,|x|<\infty, |y|\le a\right\}$. Introducing the stream (magnetization) function $g(x,y,t)$ satisfying $\boldsymbol{j}=\bar{\nabla}\times g=(\partial_y g,-\partial_x g)$, where $\boldsymbol{j}$ is the sheet current density, we present first the problem formulation for the case of a nonuniform external field vanishing at infinity and zero transport current:

$$-\nabla\times\int_{\Omega}G\left(\boldsymbol{r}-\boldsymbol{r}'\right)\bar{\nabla}'\times\dot{g}\left(\boldsymbol{r}',t\right)\mathrm{d}\boldsymbol{r}'=\dot{h}_z^{\mathrm{e}}+\mu_0^{-1}\nabla\times\boldsymbol{e},$$
$$\boldsymbol{e}=e_0(|\,\bar{\nabla}\times g\,|\,/\,j_c)^{n-1}\bar{\nabla}\times g\,/\,j_c,$$
$$g\,|_{y=\pm a}=0,\quad g\,|_{t=0}=g_0(x,y),$$

where $h_z^{\mathrm{e}}(x,y,t)$ is the z-component of the applied magnetic field in the strip plane and $\dot{g}$ means $\partial_t g$. Here we assumed the power current-voltage relation $\boldsymbol{e}(\boldsymbol{j})=e_0(j\,/\,j_c)^{n-1}\boldsymbol{j}\,/\,j_c$ with a field-dependent and, possibly, spatially inhomogeneous sheet critical current density j_c which, however, becomes spatially homogeneous as $x\to\pm\infty$. Applying the Fourier transform with respect to x to the integral equation above, using the convolution theorem, and singling out the singular parts of the integral kernels, we obtain a 1D integrodifferential equation for $\dot{g}$ in the Fourier space for each wave number k,

$$k^2\int_{-a}^{a}\left\{\mathcal{A}-\frac{1}{2\pi}\ln\left(\frac{|k|}{2}|\,y-y'\,|\right)\right\}\tilde{\dot{g}}\left(k,y',t\right)\mathrm{d}y'+\int_{-a}^{a}\left\{\mathcal{B}+\frac{1}{2\pi}\frac{1}{y-y'}\right\}\partial_{y'}\tilde{\dot{g}}\left(k,y',t\right)\mathrm{d}y'=-\tilde{Z},\qquad(1)$$

where $\tilde{f}=F[f]$ denotes the Fourier transform, $Z=\dot{h}_z^{\mathrm{e}}+\mu_0^{-1}\left(\partial_x e_y-\partial_y e_x\right)$, and the functions $\mathcal{A}\left(k,y-y'\right)$ and $\mathcal{B}\left(k,y-y'\right)$ are regular.

For problems with a given transport current $I(t)$ and a nonvanishing but uniform at infinity external field $(\boldsymbol{h}^{\mathrm{e}}\to\boldsymbol{h}_\infty^{\mathrm{e}}(t)$ as $x\to\pm\infty)$, the formulation (1) in the Fourier space has to be modified and written for the difference,

$g^{\Delta} = g(x,y,t) - g^{\infty}(y,t)$, where $g^{\infty}(y,t)$ is the solution of a 1D strip problem with the same transport current, the uniform external field $\boldsymbol{h}_{\infty}^{e}(t)$ and spatially homogeneous j_c.

III. Numerical method

To solve the singular integral equation (1), for each wave number k we seek $\tilde{g}(k,y,t)$ as an expansion in Chebyshev polynomials depending on y/a and this enables efficient treatment of both kernel singularities. The interpolating expansions in scaled Hermite functions $\Psi_j(x/L)$, vanishing at infinity and satisfying $F[\Psi_j(x/L)] = L\sqrt{2\pi}(-\mathrm{i})^j \Psi_j(kL)$, are used for the approximate Fourier transform of functions and its inverse; here L is the scaling factor. A standard solver is used for integrating in time the system of ordinary differential equations for the mesh node values of g. Details of our numerical scheme, including the algorithms for linear operations on the bivariate Hermite-Chebyshev expansions (interpolation, differentiation, integration, Fourier transform), can be found in [4]. Here we present two numerical examples.

IV. Examples

A. *Inhomogeneous Strip with Transport Current*

Let the transport current be harmonic, $I = 200\sin(2\pi t/T)$ A, with $T = 0.02$ s. We assume the strip width $2a = 12$ mm, $\boldsymbol{h}^e = \boldsymbol{0}$, $g_0 = 0$, and $n = 20$ in the power $\boldsymbol{e}(\boldsymbol{j})$ relation with the critical sheet current density $j_c = j_c^0 \chi^r(\boldsymbol{r})\chi^h(h_z)$, where $j_c^0 = 23.6\,\mathrm{A/mm}$, $\chi^r = 1 + \varphi(x+a, y+a) - \varphi(x-a, y-a)$ with $\varphi(x,y) = 0.75\exp\left(-\left[2x^2 + y^2\right]/a^2\right)$ describes the spatial nonuniformity of the strip, and $\chi^h = 1/(1 + h_0^{-1}|h_z|)$ with $h_0 = 20 j_c^0$ describes the dependence on the magnetic field. Numerical simulation results are shown in Fig. 1.

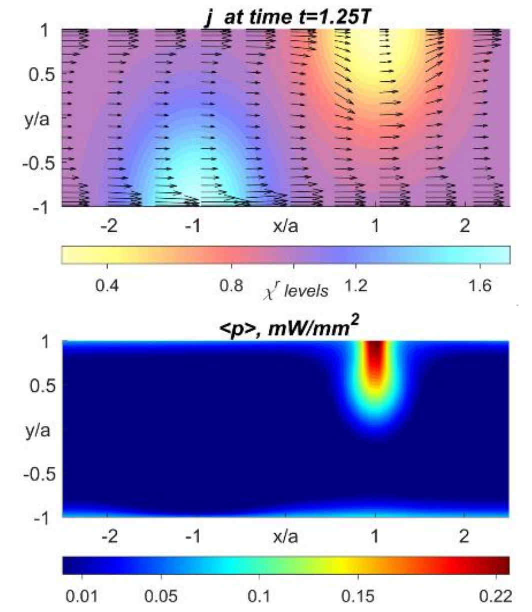

Fig. 1 Top: Sheet current density at the peak of the transport current (spatial inhomogeneity of the strip is indicated by the background colour). Bottom: Time-averaged loss power density.

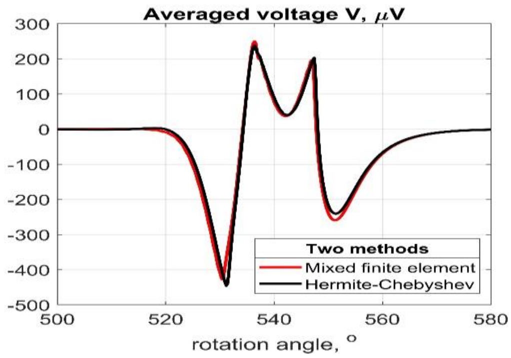

Fig. 2 The voltage induced during the second rotor rotation computed by the mixed finite element method [6] and using the spectral Hermite-Chebyshev method.

B. *Strip in a Nonuniform Field: the Dynamo Flux Pump*

A realistic 2D version of the simplified 1D superconducting dynamo flux pump "benchmark" problem [5] was solved by the finite element methods in [6,7] and also by the FFT-based method in [6]. Now we use the spectral method and compare (Fig. 2) the calculated voltages, *V(t)*, induced in the superconducting strip (dynamo stator) by the rotating permanent magnet attached to the dynamo rotor; the open contour conditions are assumed. The rate of convergence was estimated using the relative voltage deviation (in the L^1 norm), δV, from the most accurate solution for each method (Table I); these results confirm the efficiency of the new spectral method.

TABLE I. Convergence of Two Numerical Methods.

Mixed finite elements [6]			Spectral method		
N of elements	Time/cycle (min)	δV (%)	Mesh $N_x \times N_y$	Time/cycle (min)	δV (%)
1056	3.4	4.5	60×30	3.3	2.4
2180	33	1.6	90×46	31	0.9
4226	118	---	120×60	186	---

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